**Economics 130**
Elements of Mathematical Economics
Third Examination – ANSWERS
October 4, 2011

1. Find the extreme value(s) of the following function: $z = x^2 + xy + y^2 + 5$. Test whether these are minimum or maximum.

$$z = x^2 + xy + y^2 + 5$$

$$\left.\begin{array}{l} z_x = 2x + y = 0 \\ z_y = x + 2y = 0 \end{array}\right\} \quad \rightarrow \quad \boxed{\bar{x} = 0,\ \bar{y} = 0,\ \bar{z} = 5}$$

$$z_{xx} = 2, \quad z_{yy} = 2, \quad z_{xy} = z_{yx} = 1$$

Second order condition:

$$z_{xx} = 2 > 0, \ z_{xx}z_{yy} - z_{xy}{}^2 = 4 - 1 = 3 > 0. \qquad \therefore \text{minimum}$$

Note that this is unconstrained optimization. Since principal minors are all positive [2, 4], we have a minimum.

2. A two-product firm faces the following demand and cost functions below:

$$P_1 = 10 - 2Q_1 + Q_2$$
$$P_2 = 20 + Q_1 - Q_2$$
$$C = Q_1{}^2 + Q_2{}^2 + 40$$

a. Find the output levels and prices that satisfy the first order condition for maximum profit.
b. Check for second order condition for maximum.
c. What is the maximum profit?

$$R = P_1Q_1 + P_2Q_2 = [10Q_1 - 2Q_1{}^2 + Q_1Q_2] + [20Q_2 + Q_1Q_2 - Q_2{}^2]$$
$$= 10Q_1 + 20Q_2 - 2Q_1{}^2 - Q_2{}^2 + 2Q_1Q_2$$
$$\pi = R - C = [10Q_1 + 20Q_2 - 2Q_1{}^2 - Q_2{}^2 + 2Q_1Q_2] - [Q_1{}^2 + Q_2{}^2 + 40]$$
$$\pi = 10Q_1 + 20Q_2 - 3Q_1{}^2 - 2Q_2{}^2 + 2Q_1Q_2 - 40$$

2nd order condition:

$$\left.\begin{array}{l} \dfrac{\partial \pi}{\partial Q_1} = \pi_1 = 10 - 6Q_1 + 2Q_2 = 0 \\ \dfrac{\partial \pi}{\partial Q_2} = \pi_2 = 20 - 4Q_2 + 2Q_1 = 0 \end{array}\right\rangle \Leftrightarrow \left\langle \begin{array}{l} 6Q_1 - 2Q_2 = 10 \\ -2Q_1 + 4Q_2 = 20 \end{array}\right\rangle \qquad \text{two equations, two unknowns}$$

Solution: $\boxed{\bar{Q}_1 = 4, \quad \bar{Q}_2 = 7}$ and $\boxed{\bar{P}_1 = 9, \quad \bar{P}_2 = 17}$

Hessian Determinant: $H = \begin{vmatrix} -6 & 2 \\ 2 & -4 \end{vmatrix} = 20$

$H_1 < 0, H_2 > 0$. The principal minors alternate in sign $\therefore$ negative definite $\rightarrow \therefore$ Maximum!

Maximum $\bar{\pi} = 10(4) + 20(7) - 3(16) - 2(49) + 2(28) - 40 \quad \rightarrow \quad \boxed{\bar{\pi} = 50}$

3. Find the extreme values of the function $z=(5-x)y$ subject to $2x+y=20$. Use the bordered Hessian determinant test to determine if the stationary value of Z is a maximum or a minimum.

$$Z=5y-xy+\lambda(20-2x-y)$$

$$Z_\lambda=20-2x-y=0$$

$$\left.\begin{aligned}Z_x&=-y-2\lambda=0\\Z_y&=5-x-1\lambda=0\end{aligned}\right\}\rightarrow\begin{cases}-y&=2\lambda\\5-x=\lambda\end{cases}\rightarrow\frac{-y}{5-x}=2$$

$$y=2x-10$$

$$2x+(2x-10)=20\rightarrow 4x=30\rightarrow\boxed{\bar{x}=7.5,\ \bar{y}=5,\ z=-12.5}$$

For the second order condition:

$$Z_{xx}=0,Z_{yy}=0,Z_{xy}=-1,g_x=-2,g_y=-1$$

$$\left|\bar{H}\right|=\begin{vmatrix}0&g_x&g_y\\g_x&Z_{xx}&Z_{xy}\\g_y&Z_{yx}&Z_{yy}\end{vmatrix}=\begin{vmatrix}0&-2&-1\\-2&0&-1\\-1&-1&0\end{vmatrix}=\left[-(-2)(0-(1)\right]+\left[-1(2-0)\right]=-4$$

Since $\left|\bar{H}\right|=-4<0$ therefore the stationary value of Z is a $\boxed{\text{MINIMUM}}$

4. A consumer's utility function is given by $U=f(x,y)=20xy$ where x and y are levels of consumption of commodities. If $P_x=15$ per unit of x , $P_y=5$ per unit of y and his budget B is 6000 per month,

a. determine much x and y should be purchased per month to maximize U subject to his budget constraint.

b. check if the second order condition for maximum is satisfied using the bordered Hessian determinant test.

$$Z=20xy+\lambda(6000-15x-5y)$$

$$Z_\lambda=6000-15x-5y=0$$

$$Z_x=\frac{\partial Z}{\partial x}=20y-15\lambda=0$$

$$Z_y=\frac{\partial Z}{\partial y}=20x-5\lambda=0$$

$$\frac{y}{x}=\frac{15}{5}=3\rightarrow y=3x$$

$$15x+5(3x)=6000\rightarrow 30x=6000,\rightarrow\boxed{x=200\quad\rightarrow y=600}$$

$$Z_{xx}=0$$

$$Z_{yy}=0$$

$$Z_{xy}=Zyx=20$$

$$\left|H_2\right|=\begin{vmatrix}0&-15&-5\\-15&0&20\\-5&20&0\end{vmatrix}=15(100)-5(-300)=3000>0\qquad\therefore\text{MAXIMUM}$$

5. A firm has a production function given by $Q = 100K^{0.5}L^{0.5}$. The firm needs to produce 800 units of the product as required by a buyer. If $P_K = 20$ per unit and $P_L = 5$ per unit,

a. determine the levels of capital K and labor L that would <u>minimize</u> the cost of producing the required output; and

$$Z = 20K + 5L + \lambda(800 - 100K^{0.5}L^{0.5})$$

$$Z_\lambda = \frac{\partial Z}{\partial \lambda} = 800 - 100K^{0.5}L^{0.5} = 0$$

$$\left.\begin{aligned} Z_K &= \frac{\partial Z}{\partial K} = 20 - \lambda 50K^{-0.5}L^{0.5} = 0 \\ Z_L &= \frac{\partial Z}{\partial K} = 5 - \lambda 50K^{0.5}L^{-0.5} = 0 \end{aligned}\right\} \rightarrow \frac{20}{5} = \frac{\lambda 50K^{-0.5}L^{0.5}}{\lambda 50K^{0.5}L^{-0.5}} \rightarrow 4 = \frac{L}{K} \rightarrow \boxed{L = 4K}$$

Substitute to constraint:

$$100K^{0.5}L^{0.5} = 800 \rightarrow 100K^{0.5}(4K)^{0.5} = 800 \rightarrow 2K = 8 \rightarrow \boxed{K = 4,\ \ L = 16}$$

b. Calculate the total cost of producing 800 units of the product.

C=20K + 5L = 20(4)+ 5(16) = 160

6. Given the production function: $Q = AK^{0.4}L^{0.8}$.

a) $A(jK)^{.4}(jL)^{.8} = j^{1.2}AK^{.4}L^{.8} = j^{1.2}Q.$

Therefore the degree of homogeneity of the production function is equal to 1.2.

b) It exhibits increasing returns to scale since a doubling of inputs would more than double the output (2.297).

c) Output elasticities of K and L. In log terms, $\ln Q = \ln A + 0.4\ln K + 0.8\ln L$

$$\varepsilon_{q.k} = \frac{\partial Q}{\partial K}\cdot\frac{K}{Q} = 0.4 \quad \text{or} \quad \varepsilon_{q.k} = \frac{d\ln Q}{d\ln K} = 0.4$$

$$\varepsilon_{q.l} = \frac{\partial Q}{\partial L}\cdot\frac{L}{Q} = 0.8 \quad \text{or} \quad \varepsilon_{q.l} = \frac{d\ln Q}{d\ln L} = 0.8$$